SOLUTION OF THE STEADY-STATE HEAT-CONDUCTION PROBLEM FOR THIN FILAMENT HEATERS OF ARBITRARY GEOMETRY

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The temperature distribution for thin filaments heated by an electric current in vacuo is found, and simple analytical expressions are derived for calculations of the fundamental variables characterizing the thermal mode of the heater. The results of calculations according to these expressions are compared with the experimental data.

Considerable attention has been devoted to the determination of the temperature field of a heater energized by an electric current in vacuo. Due to appreciable mathematical difficulties, however, a solution has been obtained for the problem only in the case of a linear heater uncoated with insulation. A survey of the literature on the solution of the latter problem may be found in [1].

In the present article we give a solution of the system of differential equations describing, in combination with the boundary conditions, the steady-state temperature field of a filament heater of arbitrary geometry. In solving the problem we proceed from the following physical picture of the thermal energy distribution in the heater.

The heater consists of a metal filament (core of the heater), the surface of which is coated with a thin insulation layer. An electric current of density j passes along the heater core. The Joule heat released by this process is transmitted from the core to the outer surface of the insulation by heat conduction. From the outer surface of the insulation, thermal energy is radiated into vacuum according to Lambert's law [2]. Simultaneously, due to self-irradiation of the heater and the emission of radiation from the surrounding surfaces (wall of the vacuum chamber and shields), a radiative flux of density E_{inc-1} can impinge on the heater surface.

The temperature field of a filament heater is most simply determined in the coordinate system $\{r, \varphi, z'\}$. The coordinate axis coincides with the axis of the heater and, depending on the construction of the latter, can represent either a helical line (for a helical filament) or a set of straight-line segments in different planes (for a folded filament). The coordinates φ and r describe the polar angle and distance in the plane of the filament cross section at the point z'. Consequently, the coordinate system $\{r, \varphi, z'\}$ is curvilinear and nonorthogonal. Only in the case of a linear filament does this coordinate system coincide with a conventional cylindrical coordinate system. The heat-conduction equations for the heater core and the insulation coating, written in the curvilinear coordinate system $\{r, \varphi, z'\}$, have the following form for constant values of λ and ν:

$$\frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1}^{3}\frac{\partial}{\partial x'_\alpha}\left(\sqrt{g}\,g^{\beta\alpha}\frac{\partial T}{\partial x'_\beta}\right)+\frac{j^2}{\lambda}\rho(T)=0; \tag{1}$$

$$\frac{1}{\sqrt{g}}\sum_{\alpha,\beta=1}^{3}\frac{\partial}{\partial x'_\alpha}\left(\sqrt{g}\,g^{\beta\alpha}\frac{\partial \vartheta}{\partial x'_\beta}\right)=0, \tag{2}$$

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where

$$x'_1 = r; \quad x'_2 = \varphi; \quad x'_3 = z'; \tag{3}$$

$$g = r^2\left[1 + r^2\left(\frac{d\bar{\alpha}_2}{dz'}\right)^2 - r^2k^2\sin^2\varphi - 2rk\cos\varphi - \sigma^2r^2\right]; \tag{4}$$

$$g'' = 1; \quad g^{21} = g^{12} = 0; \quad g^{13} = g^{31} = 0; \quad g^{32} = g^{23} = \frac{r^2\sigma}{g};$$

$$g^{33} = \frac{r^2}{g}; \quad g^{22} = \left[1 + r^2\left(\frac{d\bar{\alpha}_2}{dz'}\right)^2 - r^2k^2\sin^2\varphi - 2rk\cos\varphi\right]\frac{1}{g}. \tag{5}$$

$\sigma(z')$ is the torsion of the axial line of the filament, k is its curvature, and $\bar{\alpha}_2$ is the unit vector normal to the filament axis.

For thin filaments, for which the values of r are not greater than 0.1 cm and the torsion σ, curvature k, and quantity $|d\bar{\alpha}_2/dz'|$ are clearly less than ten, it is possible in the expressions for g^{22} and g to neglect by comparison with unity terms containing r in the first and higher powers, i. e., to assume that $g \simeq r^2$ and $g^{22} \simeq 1/r^2$. Also, for a small filament thickness the temperature variation along the radius of the heater core can be disregarded as a small quantity. The temperature variation along the polar angle φ, on the other hand, is of no interest and can also be neglected. Accordingly, Eq. (1) must be averaged over the variables r and φ, and Eq. (2) over φ. Under these conditions Eqs. (1) and (2) reduce to the form

$$\frac{d^2T}{dz'^2} + j^2\frac{\rho(T)}{\lambda} = \frac{2}{\lambda R_1}q(z'); \quad 0 < z' < l; \tag{6}$$

$$\frac{\partial^2\vartheta}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial\vartheta}{\partial r} + \frac{\partial^2\vartheta}{\partial z'^2} = 0; \quad 0 < z' < l; \quad R_1 < r < R_2, \tag{7}$$

where $q(z')$ is the heat flux from unit lateral surface of the core.

The solution of the differential equations (6)-(7) must satisfy the boundary conditions

$$\vartheta(R_1, z') = T(z'); \tag{8}$$

$$q(z') = -\nu\left(\frac{\partial\vartheta}{\partial r}\right)_{r=R_1}; \tag{9}$$

$$-\nu\left(\frac{\partial\vartheta}{\partial r}\right)_{r=R_2} = \sigma_0\varepsilon_1\vartheta^4(R_2, z') - \varepsilon_1 E_{\text{inc-}1}(z'); \tag{10}$$

$$T(0) = T(l) = T_0; \tag{11}$$

$$\vartheta(r, 0) = \vartheta(r, l) = T_0. \tag{12}$$

We need to augment these equations with an equation determining the filament current I, which enters into Eq. (6) as an unknown parameter:

$$UI = j^2\int_V \rho(T)\,dV, \tag{13}$$

where $j = I/\pi R_1^2$ and V is the volume of the core.

It is impossible in practice to solve the system of equations (6)-(13) analytically, on account of the nonlinearity of boundary condition (10). Moreover, we do not know the function $E_{\text{inc-}1}(z')$ in Eq. (10). In order to evaluate it we need to solve the system of radiation integral equations. However, these difficulties can be surmounted as follows.

The system of equations (6)-(13) does not differ formally in any way from the system of equations describing the temperature distribution in a linear filament in a cylindrical coordinate system. It may be inferred on this basis that the behavior of the temperature distribution along a thin filament radiating in vacuum will be approximately the same irrespective of its geometric configuration. This inference enables us to use the results of experimental measurements of the temperature fields of thin linear filaments in solving the system of equations (6)-(13). A representative experimental curve, borrowed from [3], for the temperature distribution over the surface of a thin filament is shown in Fig. 1. It is clearly seen that the major part of the filament surface has a constant temperature ϑ_m. The temperature of the filament varies from ϑ_m to T_0 over a small part l' of the filament length ($l' \ll l/2$). This temperature distribution curve can be described analytically by a trigonometric series:

TABLE 1. Comparison of the Calculated Values of the Temperature ϑ_m and Length l' with the Experimental Data

Heater type	U, V	I, A	ϑ_m, °K(experimental)	ϑ_m,°K(theoretical)	l', cm(theoretical)
White	8,70	0,478	1196	1177	1,84
	13,6	0,580	1429	1438	1,60
	17,4	0,654	1595	1606	1,45
Black	15,9	0,884	1170	1145	0,99
	27,7	1,178	1442	1428	0,79
	36,7	1,374	1600	1587	0,69

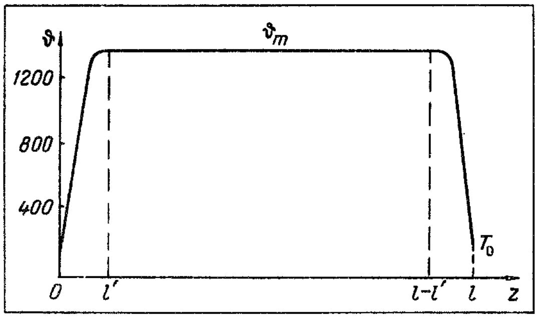

Fig. 1. Representative distribution of temperature ϑ (°K) along a thin filament.

$$F(z',\ l',\ \vartheta_m) = \sum_{n=1}^{\infty} A_n \sin k_n z' + T_0, \tag{14}$$

in which

$$\vartheta_m = \frac{T_0}{\cos P_1 l'};\quad k_n = \frac{2n-1}{l}\pi;$$

$$A_n = \frac{4P_1^2\left[T_0 - \vartheta_m \cos k_n l'\right]}{l k_n\left(k_n^2 - P_1^2\right)};\quad P_1 = \frac{j}{\pi}\sqrt{\frac{a}{\lambda}}. \tag{15}$$

Consequently, using the experimental results of measurements of the surface temperature of thin linear filaments, we found the distribution function for the temperature over the surface of a thin filament up to a constant l'. This function can be used as a new boundary condition in place of the nonlinear boundary condition (10), putting

$$\vartheta(R_2,\ z') = F(z',\ l',\ \vartheta_m). \tag{16}$$

For the determination of the unknown constant l' we use Eq. (13), and we calculate the filament current by means of the nonlinear boundary condition (10), integrating it first over the filament length. This interchange of the boundary conditions makes it possible to find a solution of Eq. (6)-(7) in the form of trigonometric series subject to the condition that $\rho(T)$ is a linear function of the temperature of the heater core ($\rho = aT + b$):

$$T = \frac{\rho_0}{a}\left[\frac{\cos q\left(z' - \frac{l}{2}\right)}{\cos q\frac{l}{2}} - 1\right] + \frac{2}{\lambda R_1}\sum_{n=1}^{\infty}\frac{a_n}{k_n^2 - q^2}\sin k_n z' + T_0; \tag{17}$$

$$\vartheta = \sum_{n=1}^{\infty}\left[b_n L_0(k_n r) + c_n K_0(k_n r)\right]\sin k_n z' + T_0, \tag{18}$$

where $\rho_0 = aT_0 + b$; $q = \sqrt{a/\lambda}$, and $L_0(k_n r)$ and $K_0(k_n r)$ are modified Bessel functions of the first and second kind. The constants a_n, b_n, and c_n are readily evaluated from the boundary conditions (8), (9), and (16). Equation (13) for the determination of l' is reduced to the following by the substitution of T from (17) and summation of the series:

$$\left[\frac{U(1-\alpha)}{jal} - \frac{b}{a}\right]\cos x - \frac{2T_0}{lP_1}\sin x = T_0 - \frac{2T_0}{lP_1}x, \tag{19}$$

where

$$x = l'P_1,\quad \alpha = \frac{\lambda R_1^2 q^2 \ln\frac{R_2}{R_1}}{2\nu}. \tag{20}$$

A singular feature of the resulting solution is that the temperature distribution function for the filament does not explicitly contain parameters characterizing the geometry of the filament or the heat-transfer conditions at its outer surface (i. e., the angular emission coefficients, emissivity of the surface, etc.). All of these quantities enter into the current equation. Their influence on the filament temperature is therefore accounted for only in terms of the current, which enters into the temperature distribution function as a parameter.

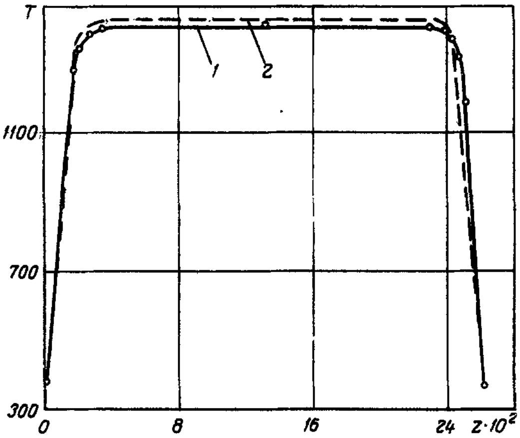

Fig. 2. Distribution of temperature T (°K) over length z' (m) of a thin tungsten filament: 1) according to experimental data [3]; 2) calculated from Eqs. (14)-(15) and (19).

We conclude with some simple relations for calculating the fundamental variables characterizing the thermal mode of the filament; these relations were derived by the approximate summation of the trigonometric series entering into (17)-(18).

The heat flux across the core cross section at the ends of the filament (annular losses) is

$$W_{\kappa} = 2\lambda \left(\frac{dT}{dz'} \right)_{z'=0} \pi R_1^2 \approx \frac{2\sqrt{a\lambda} I \vartheta_m \sin x}{\pi (1-\alpha)}. \tag{21}$$

The maximum temperature of the heater core is

$$T_m = T \left(\frac{l}{2} \right) \approx \frac{1}{1-\alpha} \left[\vartheta_m + \alpha \frac{b}{a} \right]. \tag{22}$$

The radiative self-flux from the heater surface is

$$Q_{\text{self-1}} = 2\pi R_2 \varepsilon_1 \sigma_0 \int_0^l \vartheta^4 (R_2, z')\, dz' \approx 2\pi R_2 \varepsilon_1 \sigma_0 \vartheta_m^4 \frac{1}{P_1} \left(l P_1 - \frac{5}{4} x \right) \tag{23}$$

Although the solution presented here was derived for insulation-coated filaments, it can also be used to describe the temperature field of uncoated thin filaments. For this purpose it is required to put $R_2 = R_1$ in all the equations.

For an experimental verification of the relations derived above we compare the results of calculations of ϑ_m and l' according to (15) and (19) with the experimental data. For the latter we use the results of measurements of the surface temperature of heaters coated with white and black insulation [3]. For the heaters with a white coating the emissivity of the coating surface is 0.2 or 0.3, and for the heaters with a black coating it is 0.6 to 0.8. The geometrical dimensions of the white and black heaters were identical: $l = 0.255$ m; $R_1 = 35.9 \cdot 10^{-6}$ m; $R_2 = 116.8 \cdot 10^{-6}$ m. For each type the measurements were performed at three values of the filament voltage U, so that three values of the current I and temperature ϑ_m were obtained. The results of these measurements are summarized in Table 1.

Values of ϑ_m and l' calculated according to Eqs. (19) and (15) are also given in Table 1. Experimental values for l' are not given in [3], but in that paper experimental curves are given for the temperature distribution over the length of black and white heaters, from the form of which it may be inferred that the value of l' varies between the limits from 2 to 1.5 cm for white heaters and from 1 to 0.7 cm for black heaters.

The applicability of the relations derived here to uncoated filaments can be demonstrated in a sample calculation of the temperature field of an uncoated tungsten filament. Its geometrical and electrical parameters are as follows: $l = 0.263$ m; $R_1 = 35.95 \cdot 10^{-6}$ m; U = 8 V; I = 0.349 A. Graphs of the temperature field of the uncoated tungsten filament are shown in Fig. 2. Curve 1 was plotted according to the experimental data of [3], and curve 2 was calculated according to Eqs. (14)-(15) and (19).

NOTATION

I	is the electric current in the heater;
U, λ	are the filament voltage and thermal conductivity of the heater core;
ν	is the thermal conductivity of the heater insulation;
R_1, R_2	are the radii of the core and insulation coating;
T, ϑ	are the temperatures of the core and insulation coating at a point;
ρ	is the resistivity of the heater core;
T_0	is the temperature of the heater ends;
l	is the length of the heater.

LITERATURE CITED

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